Math Circle

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What is the biggest building you've been in?

Remember primes?

- A prime number is a number which only has two factors: 1 and itself.
- The only two numbers that divide 3 are 1 and 3, so 3 is a prime.
- Since 1 only has ONE factor, we don’t consider it a prime.

The first primes

- Here is a list of all prime numbers between 1 and 100.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

- How many are there?

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The next primes

- Here is a list of all primes between 100 and 200.

101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199

- How many are there?

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More primes #enough

- Here is a list of all primes between 200 and 300.

211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293

- How many are there?

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Now your turn

- Is the number of primes between 300 and 400 more or less than the number in any of your answers

- Can you make a guess (or in mathspeak, a conjecture) without actually computing all of them?

Checking yourself

- Here is a list of primes between 300 and 400.

307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397

- How many are there? Was your conjecture correct?

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How many primes are there?

- Can you make a more general conjecture about number of primes between any two given numbers?
- Tell me some!!
- For example, what can you say about the number of primes between n and n + 100 as we pick n to be a larger and larger number?

How many primes are there?

- Consider the following problem:

- What is the remainder when $2 * 3 * 5 + 1 = 31$ is divided by
- 2?
- 3? or
- 5?

How many primes are there?

- What about the remainder when $2 * 3 * 5 * 7 * 11 + 1$ is divided by any of 2, 3, 5, 7 or 11?

- Will this always happen? Can you argue why or why not?

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Is there a largest prime number?

Breakout Rooms

Is there a largest prime number?

- No! The list of primes numbers keeps going forever.
- Let's prove that this is true: Suppose that there is a largest prime, or the last prime, and call it P.
- Number all the prime numbers less than P as follows:

$p_1 = 2$ $p_2 = 3$ $p_3 = 5$ $p_4 = 7$ $p_5 = 11$

- If we keep going like this, then we'll eventually get to $p_m = P$ for some m.

Is there a largest prime number?

- Now consider the number

$$M = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdots p_m + 1 = 2 \cdot 3 \cdot 5 \cdots P + 1$$

- So what is the remainder when M is divided by 3?
- By 7?
- Or by any other prime?

Last prime(?)

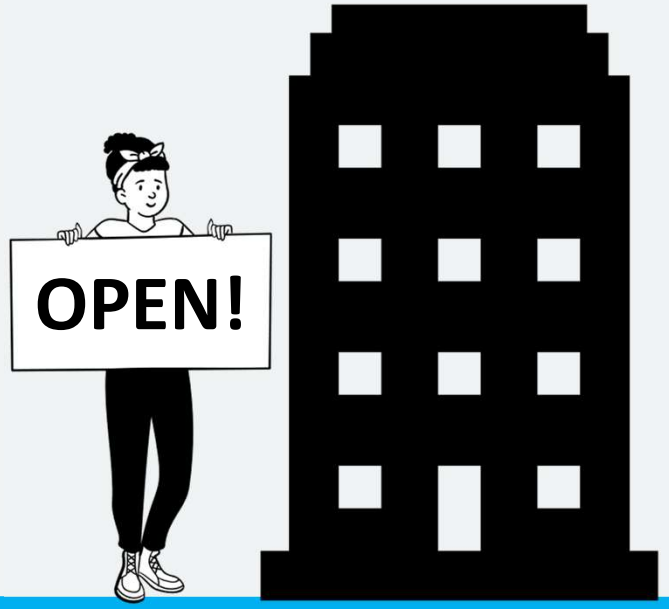
- We know that every number can be written as a product of prime numbers, for example, $24 = 2 * 2 * 3 * 3$.

(check it for yourself!)

- Think carefully about why this fact, combined with your answer to the previous question means there can be no last prime!

Hilbert's Hotels

- You are the manager of a Hilbert's Hotels.
- Unlike most hotels, yours has infinitely many rooms!
- Each room is numbered with a whole number (1, 2, 3, 4, …), and there is a room for every number.



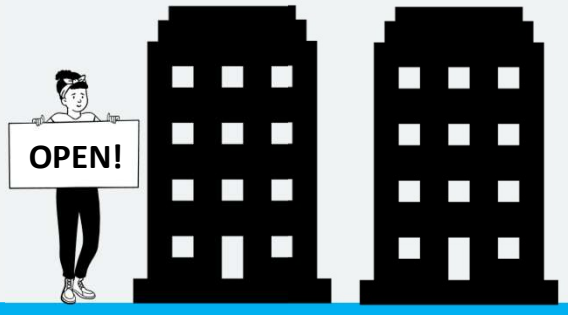
Hilbert's Hotels – Problem 1

- Imagine that every room in the hotel is occupied.
- A guest enters the hotel and asks for a room. You don't want to turn away a customer if you can help it.
- Can you find a room for this person?

- (Hint: you may ask other guests to change rooms. If you do this, you must have a way to tell each guest which room number to move to.)

Hilbert's Hotels – Problem 2

- Your hotel becomes very popular, so you open a second location, and all of its rooms are full as well.
- One cold night, the heating in your new hotel fails, and you have to find the guests somewhere else to stay.
- You'd really like to keep their business by finding them a room at your first hotel.
- How could you do this?



Hilbert's Hotels – Problem 3

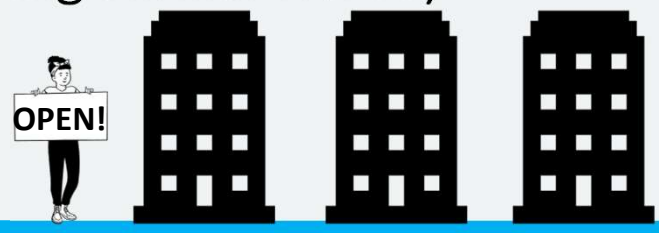
- Are there more odd numbers, or more even numbers?

- How can we think about this question in terms of our hotels?

Hilbert's Hotels – Problem 4

- With your reputation for excellent customer service, you open a third hotel.
- Call your hotels A, B, C, and imagine that each room in each hotel is occupied.
- One night, there is a water leak at both hotels B and C.
- Can you find a room for everyone at hotel A?

- (Hint: think back to problem 2. Can you do something similar here?)



Hilbert's Hotels – Challenge 1

- Are there more multiples of 2 than multiples of 3?
- There are a lot of ways to think about this question.
- Try justifying at least two different answers.
- (Hint: Could you fill up an entire hotel with just the guests in even numbered rooms? What about room numbers that are multiples of 3?)

Hilbert's Hotel – Challenge 2

- Your hotel business is really taking off! You have now opened a hotel on every street in town, and the streets are numbered 1, 2, 3, 4, … (there are infinitely many streets, one for each number). Each hotel is fully occupied tonight.

- The rival hotels have conspired against you, and cause a power failure at every one of your hotels, except the original hotel on street 1. The guests can't stay in their rooms, but you don't want to lose your good reputation with them. Can you find a room for everyone in your one hotel with power?

Hilbert's Hotel – Challenge 3

- Can you build a hotel that has more rooms than one of Hilbert's Hotels? That is, if the new hotel fails, you couldn't fit all the guests in one of Hilbert's hotels.

- (Hint: You will need to use a very different way of numbering the rooms.)